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II. Calculus of Variation

1. Euler's equation in full form.

1) Solve the variational problem $\boldsymbol{I = \int 12xy + \left(\frac{dy}{dx}\right)^2 dx = 0 \,;\, y(0) = 3, y(1) = 6.}$

Program:

```
kill(all)$
depends(f,[x,y])$
depends(y,x)$
f(x,y):=12*x*y+('diff(y,x))^2$
I1:diff(f(x,y),y)$
I2:diff(f(x,y),'diff(y,x))$
I3:diff(I2,x)$
Y1:ode2(I1-I3=0,y,x);
bc2(%,x=0,y=3,x=1,y=6);
```

Output:

```
y=x^3+%k2*x+%k1
y=x^3+2*x+3
```

2) Find the extremal of the functional $I = \int_0^{\pi/2} (y'^2 - y^2)\ dx$ given $y(0) = 2, y\left(\frac{\pi}{2}\right) = 3.$

3) Find the extremal of the functional $I = \int (y^2 + \left(\frac{dy}{dx}\right)^2 + y sechx)\, dx$.

4) Find the curve passing through $(0,0)$ and $(\pi, 0)$ along which $I = \int_0^{\pi} [\left(\frac{dy}{dx}\right)^2 + y sinx]\, dx$ an extremum.

2. Particular forms of Euler's equation

a. Equations independent of both x and y

1) **Find the extremal of the functional** $I = \int \sqrt{\left(\frac{dy}{dx}\right)^2 + 1}\; dx$ **given** $y(0) = 1, y(1) = 2.$

Program

```
kill(all)$
depends(f,[x,y])$
depends(y,x)$
dependencies$
f(x,y):=(1+('diff(y,x)))^(1/2)$
I2:diff(y,x,2)=0$
Y1:ode2(I2,y,x);
bc2(%,x=0,y=1,x=1,y=2);
```

Output:

```
y=%k2*x+%k1
 y=x+1
```

b. Equations independent of y:

2). **Find the extremal of the functional** $I = \int_{x1}^{x2} y' + x^2(y')^2\, dx$

Program

```
kill(all)$
depends(f,[x,y])$
depends(y,x)$
dependencies$
f(x,y):='diff(y,x)+x^2*('diff(y,x))^2$
I2:diff(f(x,y),'diff(y,x))$
I3:I2=k;
Y1:ode2(I3,y,x);
```

Output:

```
2*x^2*('diff(y,x,1))+1=k
 y=%c-(k-1)/(2*x)
```

1) **Find the extremal of the functional** $I = \int 3x + \sqrt{y'}\, dx$ **given** $y(1) = 5, y(2) = 7.$

3. Isoperimetric problems

1) Find the extremal of the functional $\int_0^1 (y')^2 dx$ subject to the constraint $\int_0^1 y\, dx = 1$ and having y(0)=0, y(1)=1.

Program:

```
kill(all)$
depends(f,[x,y])$
depends(g,[x,y])$
depends(y,x)$
f:('diff(y,x))^2$
g:y$
h:f+L*g$
h1:diff(h,y)$
h2:diff(h,'diff(y,x))$
h3:h1-diff(h2,x)$
h4:ode2(h3,y,x)$
h5:bc2(h4,x=0,y=0,x=1,y=1)$
h6:integrate(h5,x,0,1)$
h7:solve(rhs(integrate(h5,x,0,1))=1,L)$
h8:subst(L=h7,h5);
```

Output:

```
y=4*x-3*x^2
```

1) Find the extremal of the functional $\int_0^1 ((y')^2 + x^2)dx$ given that $\int_0^1 y\, dx = 2$ and having y(0)=0, y(1)=1.
2) Find the extremal of the functional $\int_0^\pi ((y')^2 - y^2)dx$ subject to the constraint $\int_0^\pi y\, dx = 1$ and having $(y(0) = 0, y(\pi) = 1$

IV. LINE and Multiple Integrals

1. Evaluation of the line integral with constant limits.

1) Evaluate $\int_c ydx - xdy$ along the curve $y = x^2$ from (0,0) to (1,1).

Program

```
kill(all)$
load("vect")$
x:x$
depends(y,x)$
y:x^2$
F:[-y,x]$
dx:diff(x,x)$
dy:diff(y,x)$
dr:[dx,dy]$
I1:(F.dr)$
integrate(I1,x,0,1);
```

Output: 1/3

1) Evaluate $\int_c ydx - xdy$ along the curve $y = x$ from (0,0) to (1,1).

2) Evaluate $\int_c (x + y)dx + (y - x)dy$ along the parabola $x = y^2$ from (1,1) to (4,2).

3) Evaluate $\int_c (3x + y)dx + (2y - x)dy$ along the curve $y = x^2 + 1$ from (0,1) to (3,10).

4) Evaluate $\int_c (2xy - 1)dx + (x^2 + 1)dy$ along the curve $y = x + 1$ from (0,1) to (2,3).

2. Evaluation of the double integral with constant limits

1) Evaluate $\int_0^1 \int_0^1 x^3 e^y \, dx \, dy$.

Command: integrate(integrate(x^3*%e^y,x,0,1),y,0,1);

Out put: (%e-1)/4

2) Evaluate $\int_0^1 \int_0^1 1/(x+y+1) \, dx \, dy$.

Command: integrate(integrate((1/(x+y+1)),x,0,1),y,0,1);

Output: "Is "y+2" positive, negative, or zero?"

positive;

"Is "y+1" positive, negative, or zero?"

positive;

(%o1) 3*log(3)-4*log(2)

Evaluate the following;

1) Evaluate $\int_1^2 \int_3^4 (xy + e^y) \, dx \, dy$.
2) Evaluate $\int_0^a \int_0^b (x^2 + y^2) \, dx \, dy$.
3) Evaluate $\int_0^{\pi/2} \int_0^{\pi/6} \sin(x)\cos(y) \, dx \, dy$.

3. Evaluation of the triple integral with constant limits

1) Evaluate $\int_0^a \int_0^a \int_0^a x^2 + y^2 + z^2 \; dx\, dy\, dz.$

Command: integrate(integrate(integrate(x^2+y^2+z^2,x,0,a),y,0,a),z,0,a);
Output: a^5.

1)Evaluate $\int_0^1 \int_0^2 \int_1^2 x^2yz \; dx\, dy\, dz.$

2)Evaluate $\int_1^2 \int_{-2}^{-1} \int_2^3 1/xyz \; dx\, dy\, dz.$

3)Evaluate $\int_1^2 \int_1^2 \int_1^2 (\frac{x}{y} + \frac{y}{z} + \frac{z}{x}) \; dx\, dy\, dz.$

4)Evaluate $\int_0^1 \int_0^1 \int_0^1 e^{(x+y+z)} \; dx\, dy\, dz.$

4. Evaluation of the line integral with change of variable.

1) Evaluate $\int_c xydx + x^2zdy + xyzdz$ **. where c is given by** $x = e^t, y = e^{-t}, z = t^2 \; and \;\; 0 \le t \le 1$.

Program:

```
kill(all)$
load("vect")$
x:%e^t$
y:%e^(-t)$
z:t^2$
F:[x*y,x^2*z,x*y*z]$
dx:diff(x,t)$
dy:diff(y,t)$
dz:diff(z,t)$
dt:[dx,dy,dz]$
I:(F.dt)$
integrate(I,t,0,1);
```

Output: 3/2

1) Evaluate $\int_c 3xydx - 5zdy + 10xdz$. where c is given by

$x = t^2 + 1, y = 2t^2 , z = t^3$ and $0 \le t \le 1$.

2) Evaluate $\int_c xydx + x^2zdy + xyzdz$. where c is given by

$x = e^t, y = e^{-t} , z = t^2$ and $1 \le t \le 2$. .

3) Evaluate $\int_c (3x - 2y)dx + (y + 2z)dy - x^2dz$. where c is given by

$x = t, y = 2t^2 , z = 3t^2$ and $0 \le t \le 1$.

4) Evaluate $\int_c (x + 2y)dx + (4 - 2x)dy$ around the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ in the counter clockwise.

5. Evaluation of the double integral with variable limits

1) **Evaluate:** $\int_1^4 \int_0^x xy\, dy\, dx$.

Command: integrate(integrate(x*y,y,0,x),x,1,4);

Ouput: 255/8.

2) **Evaluate:** $\int_0^{3/5} \int_0^{\sqrt{1-y^2}} dx\, dy$.

Command: assume ((y^2)<1)$

I:integrate(integrate(1,x,0,sqrt(1-y^2)),y,0,3/5);

Output: (25*asin(3/5)+12)/50)

3) **Evaluate:** $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\, dx\, dy$.

4) **Evaluate:** $\int_0^1 \int_0^{y^2} e^{\frac{x}{y}} dx\, dy$.

5) **Evaluate:** $\int_0^{\pi} \int_0^{siny} dx\, dy$.

6. Evaluation of the triple integral with variable limits

1) Evaluate: $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)dy\,dx\,dz$

Command:

integrate(integrate(integrate(x+y+z,y,x-z,x+z),x,0,z),z,-1,1);

Output: 0

2) Evaluate: $\int_{0}^{a}\int_{0}^{\sqrt{a^2-x^2}}\int_{0}^{\sqrt{a^2-x^2-y^2}}(xyz)dx\,dy\,dz$

Command: assume(a^2-x^2-y^2>0,a^2-x^2>0)$

integrate(integrate(integrate(x*y*z,z,0,sqrt(a^2-x^2-y^2)),y,0,sqrt(a^2-x^2)),x,0,a);

Output: a^6/48

3) Evaluate: $\int_{0}^{a}\int_{0}^{x}\int_{0}^{x+y}e^{x+y+z}dz\,dy\,dx$

4) Evaluate: $\int_{0}^{a}\int_{0}^{\sqrt{a^2-x^2}}\int_{0}^{\sqrt{a^2-x^2-y^2}}\frac{dz\,dy\,dx}{\sqrt{a^2-x^2-y^2-z^2}}$

5) Evaluate: $\int_{0}^{1}\int_{0}^{1-x}\int_{0}^{1-x-y}(xyz)dz\,dy\,dx$

3. Integral Theorems

1. Verification of Green's theorem.

1) Verify Green's theorem for the $\oint(x + 2y)dx + (x - 2y)dy$ where c is the curve consists of the co-ordinate axis and the line x=1,y=1.

```
kill(all)$
P(x,y):=x+2*y$
Q(x,y):=x-2*y$
s:diff(Q(x,y),x)-diff(P(x,y),y)$
Z1:integrate(integrate(s,x,0,1),y,0,1);
depends(x,y)$
F:[P(x,y),Q(x,y)]$
x(x):=1$
dx:diff(1,x)$
dy:diff(y,y)$
dt2:[dx,dy]$
H:(F.dt2)$
H1:subst(x=1,H)$
I1:integrate(H1,y,0,1);
diff(0,x)$
diff(y,y)$
dt3:[dx,dy]$
H2:(F.dt3)$
H3:subst(x=0,H2)$
I2:integrate(H3,y,1,0);
depends(y,x)$
y(y):=0$
dx:diff(x,x)$
dy:diff(0,y)$
dt:[dx,dy]$
H4:(F.dt)$
H5:subst(y=0,H4)$
I3:integrate(H5,x,0,1);
depends(y,x)$
dx:diff(x,x)$
```

```
dy:diff(1,y)$
dt1:[dx,dy]$
H6:(F.dt1)$
H7:subst(y=1,H6)$
I4:integrate(H7,x,1,0);
Z:I1+I2+I3+I4;
If Z=Z1 then
disp(Green’s theorem is verified)
else
disp(“Green’s theorem is not verified”)$
```

Output:

```
-1
0
1
1/2
-5/2
-1
Green’s theorem is verified.
```

1) Verify Green’s theorem for the $\oint(xy+y^2)dx+x^2dy$ where C is the closed curve Bounded by $y = x \ and \ y = x^2$.

3. Verification of Gauss divergence theorem.

1) **Verify the divergence theorem for $\vec{F} = 2xy\hat{\imath} + yz^2\hat{\jmath} + xz\hat{k}$ over the rectangular parallelepiped $0 \le x \le 1,\ 0 \le y \le 2,\quad 0 \le z \le 3$.**

Program:

```
kill(all)$
load("vect")$
F:[2*x*y,y*z^2,x*z];
div(F)$
express(%)$
I:ev(%,diff)$
H:integrate(integrate(integrate(I,x,0,1),y,0,2),z,0,3);
i:[1,0,0]$   j:[0,1,0]$    k:[0,0,1]$
S1:F.-k$
s1:subst(z=0,S1)$
I1:integrate(integrate(s1,x,0,1),y,0,2);
S2:F.-j$
s2:subst(y=0,S2)$
I2:integrate(integrate(s2,x,0,1),z,0,3);
S3:F.-i$
s3:subst(x=0,S3)$
I3:integrate(integrate(s3,y,0,2),z,0,3);
S4:F.k$
s4:subst(z=3,S4)$
I4:integrate(integrate(s4,x,0,1),y,0,2);
S5:F.j$
s5:subst(y=2,S5)$
I5:integrate(integrate(s5,x,0,1),z,0,3);
S6:F.i$
s6:subst(x=1,S6)$
I6:integrate(integrate(s6,y,0,2),z,0,3);
J:I1+I2+I3+I4+I5+I6;
if(H=J) then
disp("Gauss divergence theorem is satisfied")
else
disp("Gauss divergence theorem is not satisfied")$
```

Output:

[2*x*y,y*z^2,x*z]

33

0

0

0

3

18

12

33

"Gauss divergence theorem is satisfied"

2) Verify the divergence theorem for $\vec{F} = 4x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ over the region bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $2x + y + 2z = 6$.

3. Verification of Stokes theorem.

1) Verify the stokes theorem for $\vec{F} = (2x - y)\hat{\imath} - yz^2\hat{\jmath} - y^2z\hat{k}$ where S is the upper half of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary.

Program:

```
kill(all)$
load("vect")$
F:[2*x-y,-y*z^2,-y^2*z];
curl(F)$
express(%)$
E:ev(%,diff)$
k:[0,0,1]$
E1:E.k$
assume(x^2<1)$
I:4*integrate(integrate(E1,y,0,sqrt(1-x^2)),x,0,1);
depends([x,y,z],t)$
x:cos(t)$
y:sin(t)$
z:0$
F:[2*x-y,-y*z^2,-y^2*z]$
dx:diff(x,t)$
dy:diff(y,t)$
dz:diff(z,t)$
dr:[dx,dy,dz]$
F1:F.dr$
I1:integrate(F1,t,0,2*%pi);
if(I=I1) then
disp("Stokes theorem is verified")
else
disp("Stokes theorem is not verified")$
```

Output:

```
[2*x-y,-y*z^2,-y^2*z]
%pi
 %pi
"Stokes theorem is verified"
```

2) Verify the stokes theorem for $\vec{F} = y^2\hat{\imath} + xy\hat{\jmath} - xz\hat{k}$ where S is the hemisphere $x^2 + y^2 + z^2 = a^2, z \geq 0$.

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